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A resonant wave theory

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Analysis is used to show that a solution of the Navier–Stokes equations can be computed in terms of wave-like series, which are referred to as waves below. The mean flow is a wave of infinitely long wavelength and period; laminar flows contain only one wave, i.e. the mean flow. With a supercritical instability, there are a mean flow, a dominant wave and its harmonics. Under this scenario, the amplitude of the waves is determined by linear and nonlinear terms. The linear case is the target of flow-instability studies. The nonlinear case involves energy transfer among the waves satisfying resonance conditions so that the wavenumbers are discrete, form a denumerable set, and are homeomorphic to Cantor's set of rational numbers. Since an infinite number of these sets can exist over a finite real interval, nonlinear Navier–Stokes equations have multiple solutions and the initial conditions determine which particular set will be excited. Consequently, the influence of initial conditions can persist forever. This phenomenon has been observed for Couette–Taylor instability, turbulent mixing layers, wakes, jets, pipe flows, etc. This is a commonly known property of chaos.

1. Introduction

Many physical phenomena, such as fluid flows and quantum mechanics, have irregular wave-like behaviour. The dynamics of energy transfer between waves may be key in understanding how irregular wave-like behaviour can arise. Reynolds (1895) may have been the first to grasp this insight by calculating the energy transfer between the mean flow and a train of sinusoidal waves. Landahl (1967) observed that pressure correlation in jets and free shear layers produced strong evidence of wave-like fluctuations. He argued that the random component of turbulence excites relatively coherent and long-lived waves, which are the most lightly damped eigenmodes of the linear instability problem. He also suggested that a formal solution of turbulence can most easily be obtained through an expansion in terms of the eigenfunctions of the homogeneous problem. Along the same line of reasoning, Lighthill (1969) proposed that large eddies can be thought of as arising from the primary instabilities of the mean flow profile; he advanced the concept of wave-packet dynamics to describe the energy transfer between turbulent eddies and the mean flow. More recently, taking advantage of modern computing power, instabilities of mixed convection and Couette–Taylor flows have been explicitly determined using a wave expansion method (Yao & Ghosh Moulic 1994, 1995*b*; Yao 1995; Ghosh Moulic & Yao 1996). This is the first theoretical verification of multiple solutions for the Couette–Taylor instability, which was systematically studied by Coles (1965). The method, which is similar to Prigogine's (1962) interaction representation of Liouville's equation in non-equilibrium classical and quantum statistical mechanics, is fully nonlinear.

The principal purpose of this paper is to develop a wave theory based on the computation of Yao & Ghosh Moulic (1994, 1995*a*). This wave theory captures some fundamental properties shared by most solutions to the Navier–Stokes equations. In § 2, a solution of the Navier–Stokes equations is expressed in terms of wave-like series. For abbreviation, the wave-like series will be referred to as waves in this paper. The existence of wave solutions is guaranteed by the completeness of the expansion functions. The amplitude of the waves is determined by linear and nonlinear terms. The linear term is the subject of extensive study of linear instability. In § 3, it is shown that the nonlinear terms represent the discrete energy transfer among the waves that satisfy resonance conditions. A substantial amount of energy can only be transferred resonantly, and not by forced vibration. The idea presented in this paper is not new; in fact, it was shown independently by Stokes in 1847 and by Poincaré in 1892 that a long-time solution of a nonlinear differential equation is determined by the resonance structure of nonlinear terms (Kevorkian & Cole 1981). We extend it to partial differential equations.

More recently, Foias & Saut (1991) used the asymptotic expansion to construct a normal form for the Navier–Stokes equations with potential body forces. The linear term of the normal form is the Stokes operator and the nonlinear operator involves terms corresponding to resonances in the spectrum of the Stokes operator. Chen, Goldenfield & Oono (1994) demonstrated that renormalization group theory could be used to understand asymptotic expansion methods in a unified fashion. Amplitude equations obtained by the reductive perturbation methods are renormalization group equations.

The reason why the Navier–Stokes equations can have multiple solutions is that excited discrete wavenumbers form a denumerable set and an infinite number of these sets can exist over a finite real interval as warranted by Canter's number theory (Courant & Robbins 1941). The initial or upstream condition determines the specific set of wavenumbers that will be excited. Numerical solutions and experiments with viscous flows are discussed in § 4. In § 5, some fundamental solution properties are summarized.

2. Existence of wave solutions

The following derivation is coordinate independent so that the result is general. As an example of the approach, let us consider a parallel one-dimensional mean flow and study the evolution of this flow to three-dimensional disturbances of arbitrary waveform. The disturbance is initially assumed to have a continuous spectrum and is represented by a Fourier integral. With an appropriate non-dimensionalization, the continuity and momentum equations are

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{u} &= 0, \\ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{u}, \end{aligned}\right\} \tag{1}$$

where Re is the Reynolds number, $\boldsymbol{u} = (u, v, w)$ are the velocity components in the (x, y, z) directions respectively, p is the pressure and t is the time. The coordinate y can be a similarity variable for a slowly growing thin layer, or the normal coordinate for a fully developed channel flow.

Equation (1) admits the steady parallel-flow solution $u = U(y)$, $v = w = 0$ for properly defined boundary conditions. We study the evolution of this parallel flow by superimposing a disturbance on the basic flow, and writing the disturbed velocity as

$$\boldsymbol{u} = (u, v, w) = (U(y) + u', v', w'), \tag{2}$$

where the primes denote disturbances whose magnitudes are not necessarily small. Substitution of (2) into (1) leads to a system of equations for the disturbance. The disturbance is expressed as Fourier integrals over all possible continuous wavenumbers. Thus, the disturbance velocity is written as

$$\boldsymbol{u}'(\boldsymbol{x}, y, t) = \int_{-\infty}^{\infty} \hat{\boldsymbol{u}}(\boldsymbol{k}, y, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{k}, \tag{3}$$

where $\boldsymbol{x} = (x, z)$, and $\boldsymbol{k} = (k_x, k_z)$. The continuity equation for the disturbance in Fourier space becomes

$$D[\boldsymbol{k}, \hat{\boldsymbol{u}}] = \mathrm{i}(k_x \hat{u} + k_z \hat{w}) + \frac{\partial \hat{v}}{\partial y} = 0, \tag{4a}$$

and the momentum is

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \boldsymbol{L}(\boldsymbol{k}, \hat{\boldsymbol{u}}, \hat{p}; Re) = \hat{\boldsymbol{N}}, \tag{4b}$$

where $\boldsymbol{L}$ is the vector of linear terms. The nonlinear terms in Fourier space are defined exactly by the integrals

$$\hat{\boldsymbol{N}}(\boldsymbol{k}, y, t) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}. \tag{5}$$

The linear instability is studied by assuming an infinitesimally small disturbance, neglecting the nonlinear integrals, and expressing the Fourier components of the disturbance quantities in separable form as

$$\hat{\boldsymbol{u}}(\boldsymbol{k}, y, t) = \widetilde{\boldsymbol{U}}(\boldsymbol{k}, y) \mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})t}, \tag{6}$$

where $\widetilde{\boldsymbol{U}}$ is the eigenfunction, $\omega(\boldsymbol{k}) = \omega^R + \mathrm{i}\omega^I$ is the complex frequency for the wavenumber $\boldsymbol{k}$ and the superscripts '*R*' and '*I*' denote the real and imaginary parts. The linearized disturbance equations become

$$D[\boldsymbol{k}, \widetilde{\boldsymbol{U}}] = 0, \quad \boldsymbol{L}(\boldsymbol{k}, \widetilde{\boldsymbol{U}}, \tilde{p}; Re) = \mathrm{i}\omega \widetilde{\boldsymbol{U}}. \tag{7}$$

Equations (7) with the appropriate boundary conditions form an eigenvalue problem for the complex frequency ω, with the disturbance being linearly unstable for $\omega^I > 0$. For a bounded domain, the eigenfunctions form a complete set (DiPrima & Mabetler 1969).

The nonlinear solution of (4) is sought by the expansion of the eigenfunctions as

$$\hat{\boldsymbol{u}}(\boldsymbol{k}, y, t) = \sum_{m=1}^{\infty} A_m(\boldsymbol{k}, t) \widetilde{\boldsymbol{U}}_m(\boldsymbol{k}, y), \tag{8}$$

where $\widetilde{\boldsymbol{U}}_m$ is the eigenfunction corresponding to the mth eigenvalue ω_m, and A_m is a time-dependent amplitude-density function. The eigenvalues, ω_m, are ordered so that $\omega_1^I \geqslant \omega_2^I \geqslant \omega_3^I \geqslant \cdots$. Thus, the first eigenvalue represents the least stable or the most unstable mode. The eigenfunctions are normalized so that $\int \left[|\tilde{u}_m|^2 + |\tilde{v}_m|^2 + |\tilde{w}_m|^2 \right] \mathrm{d}y = 1$. We normalize the adjoint eigenfunctions $\boldsymbol{U}^\dagger$ by requiring the vector inner product

$$\langle \boldsymbol{U}^{\dagger}, \widetilde{\boldsymbol{U}} \rangle = \langle u^{\dagger}, \tilde{u} \rangle + \langle v^{\dagger}, \tilde{v} \rangle + \langle w^{\dagger}, \tilde{w} \rangle = \delta_{ij}, \tag{9}$$

where δ_{ij} is the Kronecker delta. The inner product of two scalar functions is defined as $\langle u^{\dagger}, \tilde{u} \rangle = \int u^{\dagger^*} \tilde{u}\, \mathrm{d}y$, where the asterisk denotes a complex conjugate. Projection of (4) on $\boldsymbol{U}_m^{\dagger}$ yields, on using (9) and the continuity equation, the following system of coupled nonlinear integro-differential equations for the amplitude density functions:

$$\frac{\mathrm{d}A_m}{\mathrm{d}t} + \mathrm{i}\omega_m A_m = \langle \boldsymbol{U}_m^{\dagger}, \hat{\boldsymbol{N}} \rangle = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} I(\boldsymbol{k}, m, m_1, m_2, t), \tag{10}$$

where $I(\boldsymbol{k}, m, m_1, m_2, t) = \int_{-\infty}^{\infty} b(\boldsymbol{k}, \boldsymbol{k}_1, m, m_1, m_2) A_{m_1}(\boldsymbol{k}_1, t) A_{m_2}(\boldsymbol{k} - \boldsymbol{k}_1, t)\, \mathrm{d}\boldsymbol{k}_1$, and the b are time-independent interaction coefficients which depend on the eigenfunctions and adjoint eigenfunctions. In general, the solution of (10) must be obtained by a numerical method. Stable numerical solutions suggest the existence of wave solutions for the Navier–Stokes equations. The above derivation is a simple consequence of the elementary functions used in the expansion to find the solutions of the Navier–Stokes equations being linearly independent and complete. Therefore, the derivation is not limited to a parallel mean flow and can also be extended to other nonlinear partial differential equations. In the next section, we show that the solution to (10) is not unique.

It is worth pointing out that the above derivation is equally valid if using Chebyshev polynomials for the expansion in the y-direction. Such an expansion is known as the Fourier–Chebyshev spectral method and is a popular method for direct numerical simulation of turbulence (Kim & Moin 1986). The numerical solutions of (10) for mixed convection and Couette–Taylor flows show that a supercritical instability contains a mean flow as well as a dominant wave and its harmonics. Since $\boldsymbol{k} = 0$ is included in the summation of (10), the mean flow can be interpreted as a wave, participating in the nonlinear wave interaction. For laminar flows, no wave other than the mean flow can persist.

The wave structure along the y-direction is an intrinsic property of the eigenfunctions. Each eigenfunction is formed by a group of waves since it is the sum of expansion functions. The dynamic structure of the mean flow determines the eigenfunctions, which differ for different flows. In a bounded domain, the eigenvalues are discrete and complete. Since the energy transfer is via resonance, as we show below the wavenumbers are therefore discrete. Even though a continuous spectrum in the y-direction possibly exists, in theory, for an unbounded domain, no energy can transfer nonlinearly to it. We may also interpret the eigenfunctions as modules for flow visualization, such as bursting and sweeping of hairpin vortices, merging or tearing of vortices, flow separations, etc. When the amplitude of an eigenfunction increases, its associated visual effect appears. Therefore, energy transfer from eigenfunctions to eigenfunctions, or from a group of waves to another group causes a visual change of flow patterns.

3. Nonlinear energy transfer

The physics of nonlinear terms is studied by expanding the amplitude density function in a series. The maximum amplification rate predicted by linear instability theory is used as the expansion parameter $\varepsilon = |\omega_1^I|$ (Yao & Ghosh Moulic 1995*a*).

The amplification rate predicted by linear theory for the mth eigenmode of the wave with the wavenumber $\boldsymbol{k}$ be expressed in terms of ε as $\omega_m^I(\boldsymbol{k}) = \varepsilon a_m(\boldsymbol{k})$. This makes a_m a constant of order one. It is convenient to write the coefficients in (8) as

$$A_m(\boldsymbol{k}, t) = \widetilde{A}_m(\boldsymbol{k}, t)\mathrm{e}^{-\mathrm{i}\omega_m^k(\boldsymbol{k})t}, \tag{11}$$

where ω_m^R is the (real) frequency predicted by linear theory, so that (10) takes the form

$$\frac{\mathrm{d}\widetilde{A}_m}{\mathrm{d}t} = \varepsilon\, a_m\widetilde{A}_m + \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\widetilde{I}(\boldsymbol{k}, m, m_1, m_2, t), \tag{12}$$

where

$$\widetilde{I} = \int_{-\infty}^{\infty} b(\boldsymbol{k}, \boldsymbol{k}_1, m, m_1, m_2)\widetilde{A}_{m_1}(\boldsymbol{k}_1, t)\widetilde{A}_{m_2}(\boldsymbol{k} - \boldsymbol{k}_1, t)\mathrm{e}^{\mathrm{i}\Omega_{3W}t}\,\mathrm{d}k_1,$$

and

$$\Omega_{3W} = \omega_m^R(\boldsymbol{k}) - \omega_{m_1}^R(\boldsymbol{k}_1) - \omega_{m_2}^R(\boldsymbol{k} - \boldsymbol{k}_1).$$

The amplitude density function $\widetilde{A}_m$ is expanded in a perturbation series as

$$\widetilde{A}_m(\boldsymbol{k}, t) = \varepsilon A_{m,1}(\boldsymbol{k}, t, T_1, T_2) + \varepsilon^2 A_{m,2}(\boldsymbol{k}, t, T_1, T_2) + \varepsilon^3 A_{m,3}(\boldsymbol{k}, t, T_1, T_2) + \cdots, \tag{13}$$

where $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow time scales. Substituting (13) into (12) and following the standard procedure of the multiple-timing method results in the evolution equation of the leading-order amplitude density function:

$$\frac{\partial A_{m,1}(\boldsymbol{k}, T_1)}{\partial T_1} = a_m A_{m,1} + \sum_{m_1,m_2}\widetilde{I}_3(\boldsymbol{k}, m, m_1, m_2, T_1) + \sum_{\substack{m_1,m_2,\\ m_3,m_4}}\widetilde{I}_4(\boldsymbol{k}, m, m_1, m_2, m_3, m_4, T_1) + \cdots\ . \tag{14}$$

The integrals, $\widetilde{I}_3 = \int_{-\infty}^{\infty}\delta(\Omega_{3W})\left[b + \varepsilon\tilde{b}\right]A_{m_1,1}(\boldsymbol{k}_1, T_1)A_{m_2,1}(\boldsymbol{k} - \boldsymbol{k}_1, T_1)\mathrm{e}^{\mathrm{i}\Omega_{3W}t}\,\mathrm{d}\boldsymbol{k}_1$, represent energy transfer from the waves of $\boldsymbol{k}_1$ and $\boldsymbol{k} - \boldsymbol{k}_1$ to the wave of $\boldsymbol{k}$. Since $\delta(\Omega_{3W})$ is the delta function, the integral $\widetilde{I}_3$ can be readily evaluated to show that they are resonant trios. The resonance condition that $\Omega_{3W} = 0$ is an algebraic equation for $\boldsymbol{k}_1$, and only discrete $\boldsymbol{k}_1$ satisfy it.

The multiple-timing perturbation method also indicates that the amplitude density functions are subjected to forced vibration of the quadratic nonlinear terms in the equations, but limited energy can be transferred via non-resonant forced vibration. Since the wave structure is implicitly included in the eigenfunctions, no resonance condition on the index of the m can be found explicitly. We believe that resonant conditions in the y-direction also exist.

Similarly,

$$\widetilde{I}_4 = \int_{-\pi}\int^{\infty}\delta(\Omega_{4W})\varepsilon\tilde{c}A_{m_1,1}(\boldsymbol{k}_1, T_1)A_{m_3,1}(\boldsymbol{k}_2, T_1)A_{m_4,1}(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2, T_1)\mathrm{e}^{\mathrm{i}\Omega_{4W}t}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2,$$

with $\Omega_{4W} = \omega_m^R(\boldsymbol{k}) - \omega_{m_1}^R(\boldsymbol{k}_1) - \omega_{m_3}^R(\boldsymbol{k}_2) - \omega_{m_4}^R(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) = 0$ being resonant quartets, where $\tilde{b}$ and $\tilde{c}$ are functions of the interaction constant b and frequencies. This shows that transferring energy among waves is via resonance; consequently, the wavenumber is discrete. For the quadratic nonlinearity of the Navier–Stokes equations, the resonance conditions, shown above, indicate that it requires a minimum of three waves for resonance. For other types of nonlinearity, the resonance conditions may differ slightly, but can be readily found. Phillips (1960) seems to have first recognized that significant energy can only transfer via resonance among water waves. It also is interesting to note that Zakharov's integral equation for water waves (Yuen & Lake 1982) is similar to (15), but he only considered four-wave resonance.

In the above derivation, the wave frequencies are frozen, so it is valid only for small amplitudes. For larger amplitudes, resonance can modify wave frequencies. As wave frequencies change, they can switch in or out of resonance groups (see the Appendix). This resonant procedure is too complicated to be traced analytically. It has been observed numerically in (10) for nonlinear instability with the Rayleigh or Reynolds number much larger than its critical value (Yao & Ghosh Mouli 1995; Ghosh Mouli & Yao 1996) and in experimental measurements for turbulent mixing layers (Zhou 1997). Also, Kachanov (1994) illustrated the wave resonance mechanism for K-type and C-type boundary-layer transitions. Experimentally, Breuer, Cohen & Haritonidis (1997) show that the early and late stages of boundary-layer transition to turbulence simply involve the discrete excitation of many waves via resonance. Although there is no rigorous proof that substantial energy transfer between waves is exclusively via resonance, it is supported by physical arguments as well as numerical evidence. Since resonance broadens the frequency spectrum for each wave, a time-averaging measurement will show a continuous frequency spectrum (see the Appendix).

Equation (15) contains infinitely many resonant wave groups; consequently, a long-time solution of nonlinear partial differential equations is determined by the resonance structure. The nonlinear interaction of a large number of waves can be complex. At present, it is unclear if it is possible to sort out all the wave groups for any particular flow. For a simpler nonlinear Kuramoto–Sivashinsky equation, numerical evidence of the existence of several wave groups is shown in the Appendix. In the following, we outline a few simple and commonly observed resonances:

(i) The resonance condition requires that the frequencies of waves satisfy the exact relation of the wavenumbers. If the waves are phase-locked, then the resonance condition is automatically satisfied. Two commonly observed cases are harmonic and subharmonic resonances. The harmonic resonance of three waves occurs when a wave interacts with itself and transfers energy to its first harmonic. Let us represent this symbolically as $2\boldsymbol{k} = \boldsymbol{k}+\boldsymbol{k}$. The left-hand side of the equality sign is the new wave and the right-hand side is the interacting waves. The subharmonic resonance is $\boldsymbol{k} = 2\boldsymbol{k}-\boldsymbol{k}$. This shows that the subharmonic wave must exist before this resonance can appear, since $\boldsymbol{k}$ appears on the right-hand side of the equals sign. The numerical results of Yao & Ghosh Moulin (1994) confirmed this. In reality, waves of all wavenumbers can exist, but their amplitudes may be too low to be noticeable. The harmonic and subharmonic resonances have been observed in turbulent mixing layers (Dimotakis & Brown 1976; Browand & Troutt 1980; Oster & Wygnanski 1982; Huang & Ho 1990), wakes and jets (Crow & Champagne 1971; Moore 1976).

(ii) The earliest recognized resonance group in fluid dynamics was $\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{k} - \boldsymbol{k}$. This is the self-interaction of resonant quartets that transfer energy from waves to the mean flow to balance the linear growth of waves. This resonance condition is always satisfied. This is the resonance considered by most weakly nonlinear instability theories (Stuart 1960; Watson 1960; Stewartson & Stuart 1971).

(iii) All waves can always interact nonlinearly with the mean flow. Symbolically, this is $\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{0}$ which forms a resonant trio. This is the most important one for transient flows, and linear instability is a special case for this kind of resonance. The existence of this resonance allows all waves to interchange energy indirectly via their interaction with the mean flow.

A consequence of discrete spectra is that the Navier–Stokes equations have multiple solutions, even for fully developed flows. This can be explained by Cantor's theory of sets. The wavenumbers of the discrete spectrum form a denumerable set, and infinitely many sets can exist on a finite real interval. The initial conditions determine which set will be excited in any particular situation. This means that flows above the critical Reynolds number may have multiple solutions. This phenomenon seems to have first been systematically investigated by Coles (1965) for Couette–Taylor instability, and has been observed in many turbulent flows (see references cited above).

For fully developed pipe flows, Ramaprian & Tu (1980) found two entirely different steady states under apparently identical conditions. Multiple solutions have been observed for many turbulent shear flows after reaching their self-similar states, as has been argued by George (1989). Slessor, Bond & Dimotakis (1998) show that a mixing layer at high Reynolds numbers and away from inflow boundaries is found not to be a unique function of local-flow parameters. They also show that inflow conditions influence not only large-scale motions, but also small-scale (molecular) mixing in the shear layers. This is consistent with the resonant wave theory.

4. Discussion

Coles (1965) observed that axisymmetric Taylor-vortex flow becomes unstable as the angular speed of the inner cylinder is increased further, beyond a second critical value. This instability results in a wavy-vortex flow, with azimuthally propagating waves superposed on the Taylor vortices. Coles found that the spatial structure of the wavy-vortex flow, characterized by axial and azimuthal wavenumbers, is not a unique function of the Reynolds number and boundary conditions. Different equilibrium states could be achieved at the same Reynolds number by approaching the final Reynolds number with different acceleration rates, and by rotating and then stopping the outer cylinder.

The non-uniqueness of the equilibrium state observed by Coles was subsequently observed in time-independent Taylor vortices. Snyder (1969) found that, while the wavelength at the onset of instability was unique, Taylor-vortex flows with different wavenumbers could be obtained at the same value of the Reynolds number by varying the initial conditions. He observed that there was a band of accessible wavenumbers, smaller than the band that can grow according to linear theory. Burkhalter & Koschmieder (1974) found that the range of axial wavelengths for stable Taylor-vortex flow is quite large. Benjamin (1978) observed different spatial states even in an annulus so short that only three or four vortices could be accommodated. Fenstermacher, Swinney & Gollub (1979) studied the transition to turbulence in Couette–Taylor flow and found that the different spatial states had different spectra and transition Reynolds numbers.

Tagg (1994) provides a recent comprehensive review on the large volume of work in the Couette–Taylor problem. He states in the conclusion that the Couette–Taylor system is so basic to understanding hydrodynamic stability, pattern formation, and turbulence that it is sometimes referred to as the 'hydrogen atom' of fluid mechanics. It is our opinion that a systematic study of the problem from the point of view of the resonant wave theory may shed new light on many phenomena in fluid dynamics at high Reynolds numbers.

The numerical results of Ghosh Moulic & Yao (1996) confirmed the observation of Coles and Snyder thirty years later. The selection principles of equilibrium waves outlined in their numerical results of mixed convection and Couette–Taylor instability agree with Snyder's results. They are

(i) When the initial disturbance consists of a single dominant wave within the nonlinear stable region, the initial wave remains dominant in the final equilibrium state. Consequently, for a slow starting flow, the critical wave is likely to be dominant.

(ii) When the initial condition consists of two waves with finite amplitudes in the nonlinear stable region, the final dominant wave is the one with the higher initial amplitude. If the two waves have the same initial finite amplitude, the dominant wave seems to be the one closer to the critical wave. On the other hand, if the initial amplitudes are very small, the faster growing wave becomes dominant. This principle has been demonstrated experimentally by Oster & Wygnanski (1982) for turbulent mixing layers.

(iii) When the initial disturbance has a small uniform continuous spectrum, the final dominant wave is the fastest linearly growing wave. On the other hand, if the uniform noise level is not small, the critical wave is the dominant equilibrium one. The numerical results clearly show that the dominant wave and its harmonics are discrete.

(iv) Any initial disturbance outside the accessible wavenumber or frequency range will excite its subharmonic or harmonic, whichever is inside the accessible range. The accessible range is smaller than the linear instability range and can only be determined by nonlinear analyses. The process includes a decreasing of the amplitude of one wave and increasing of the other. During the beginning of the process, the waves are not necessarily phase-locked. These types of resonance have been observed in mixing layers. The visual consequence of these resonances is the merging or tearing of vortices. The experiments also indicate that the growth of turbulent shear layers always accompanies nonlinear energy transfer to waves of lower frequencies and longer wavelengths (Huang & Ho 1990).

5. Conclusion

If the solution of nonlinear Navier–Stokes equations is expressed in terms of waves, the amplitude of the waves is determined by linear and nonlinear terms. The effects of linear terms are well-known. The nonlinear terms are responsible for transferring energy resonantly among waves. Therefore, the structure of resonance determines long-time solutions. Some of the consequences of resonance are listed below:

(i) Waves interact resonantly within their group. The resonance conditions can be satisfied only by discrete waves. Many resonance groups can coexist, but they may not be correlated. Waves can enter or leave any particular resonance group.

(ii) All waves can interact resonantly with a wave of infinite wavelength and zero frequency.

(iii) The set of excited wavenumbers depends on the initial or upstream conditions. Therefore, there are multiple solutions.

Since the solution properties of nonlinear partial differential equations outlined above are rather general, a well-designed numerical method or a phenomenological model should be consistent with them.

Appendix. One-dimensional model: Kuramoto–Sivashinsky equation

The Kuramoto–Sivashinsky equation can be written as

$$u_t + 4u_{xxxx} + \lambda(u_{xx} + uu_x) + \beta u_x = 0, \tag{A 1}$$

in the limit that $\beta \to 0$. Applying the Fourier transformation,

$$\hat{u} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

to (A 1) results in

$$\hat{u}_t = [-\mathrm{i}\omega(k) + k^2(\lambda - 4k^2)]\hat{u} - \mathrm{i}\lambda\int_{-\infty}^{\infty} k'\hat{u}(k')\hat{u}(k-k')\,\mathrm{d}k', \tag{A 2}$$

where $\omega(k) = \beta k$ is the frequency. The minimum wavenumber can be easily determined for a multi-dimensional flow, but this is not the case for the one-dimensional Kuramoto–Sivashinsky equation. We artificially select k_1 as the minimum wavenumber in order to show that resonance results in discrete waves, and set

$$\lambda = 4k_1^2 + \varepsilon, \tag{A 3}$$

$$\hat{u}(k_1) = A(k_1)\mathrm{e}^{-\mathrm{i}\omega(k_1)t}, \tag{A 4}$$

where A is the complex amplitude-density function. We introduce a slow time scale $\tau = \varepsilon t$ and expand the amplitude-density function as

$$A(t,\tau) = \varepsilon A_1 + \varepsilon^2 A^2 + \cdots\ . \tag{A 5}$$

Substituting (A 3)–(A 5) into (A 2) results in

$$\lfloor \varepsilon A_1 + \varepsilon^2 A_2 + \cdots \rfloor_{,t} + \varepsilon\lfloor \varepsilon A_1 + \varepsilon^2 A_2 + \cdots \rfloor_{,\tau} = \varepsilon k_1^2 \lfloor \varepsilon A_1 + \varepsilon^2 A_2 + \cdots \rfloor$$
$$-\mathrm{i}(4k_1^2 + \varepsilon)\int_{-\infty}^{\infty} k'[\varepsilon A_1(k') + \cdots][\varepsilon A_1(k_1 - k') + \cdots]\mathrm{e}^{\mathrm{i}\Omega(k')t}\,\mathrm{d}k', \tag{A 6}$$

where $\Omega(k') = \omega(k_1) - \omega(k_1 - k') - \omega(k')$. Collecting the first-order terms results in $A_{1,t} = 0$; consequently, $A_1 = A_1(\tau)$. Collecting the second-order terms results in

$$\left[A_{2,t} + \mathrm{i}4k_1^2\int_{-\infty}^{\infty}[1 - \delta(\Omega)]k'A_1(k')A_1(k_1 - k')\mathrm{e}^{\mathrm{i}\Omega(k_1,k')t}\,\mathrm{d}k'\right]$$
$$+\left[A_{1,\tau} - k_1^2 A_1 + \mathrm{i}4k_1^2\int_{-\infty}^{\infty}\delta(\Omega)k'A_1(k')A_1(k_1 - k')\mathrm{e}^{\mathrm{i}\Omega(k_1,k')t}\,\mathrm{d}k'\right] = 0, \tag{A 7}$$

where δ is the delta function. Ruling out the secular terms, (A 7) becomes

$$A_{2,t} = -\mathrm{i}4k_1^2\int_{-\infty}^{\infty}[1 - \delta(\Omega)]k'A_1(k')A_1(k_1 - k')\mathrm{e}^{\mathrm{i}\Omega(k_1,k')t}\,\mathrm{d}k'. \tag{A 8}$$

This is an equation for forced vibration due to the quadratic nonlinearity of the equation. It is well known from the dynamics of oscillators that comparatively little energy transfers through forced vibration modes. The function A_1 can be determined

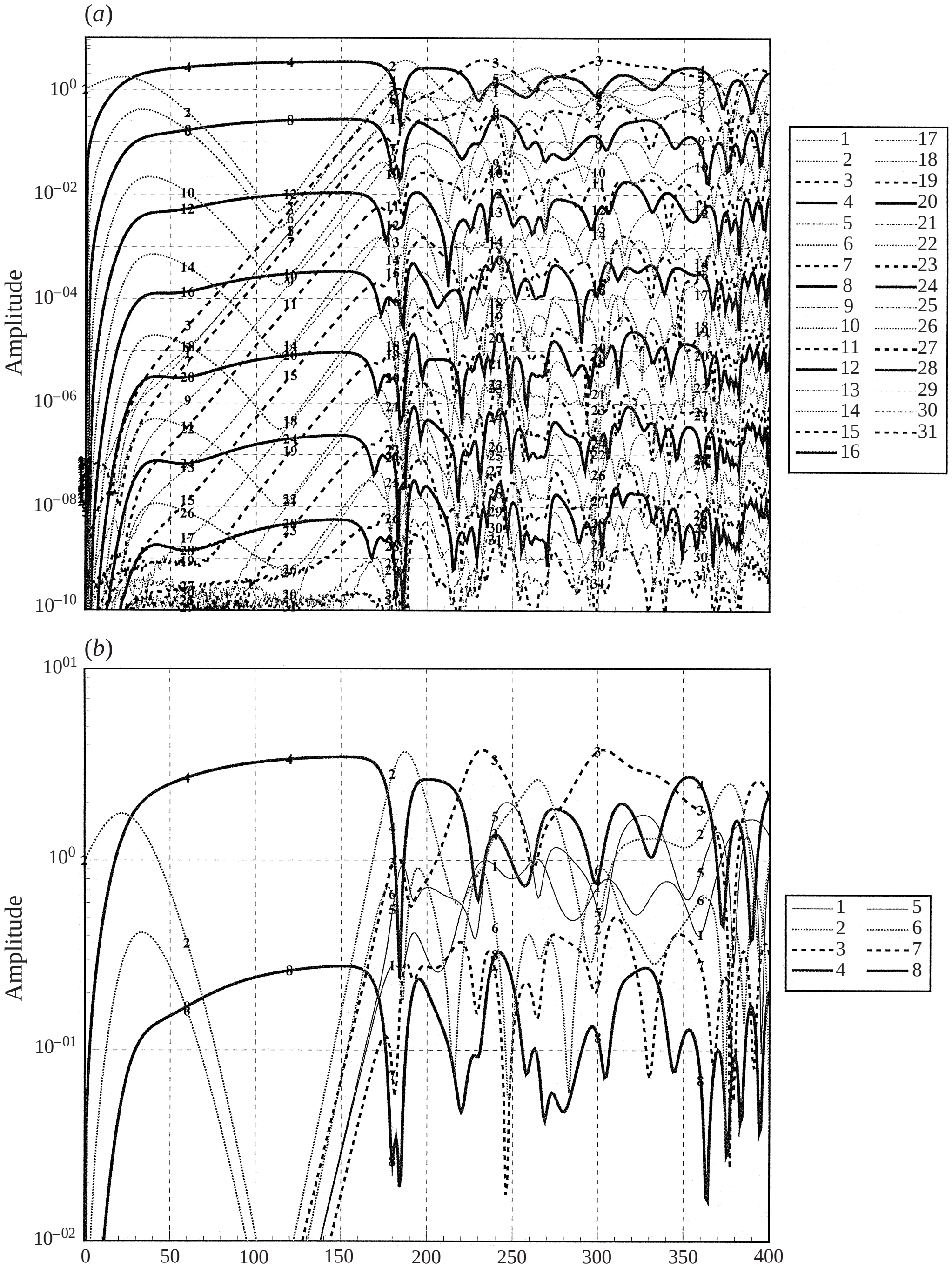

FIGURE 1(a, b). For caption see facing page.

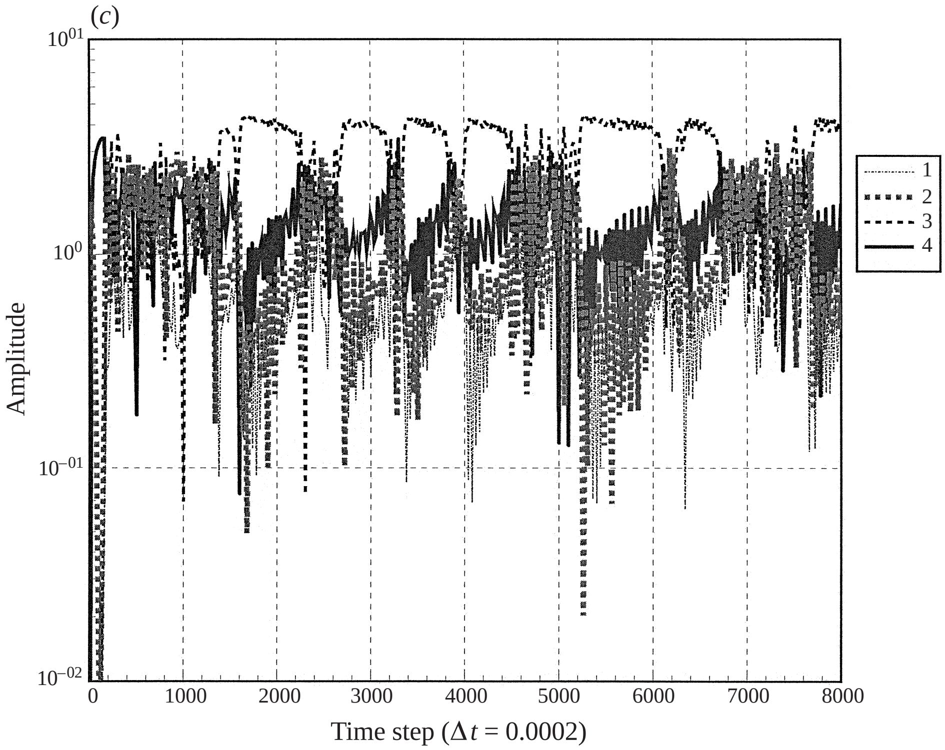

FIGURE 1. (*a*) Amplitude with initial mode: $2 = (1, 0)$. (*b*) Amplified plot of the first eight waves. (*c*) Long-time plot of the first four waves.

from the secular terms in (A 7) as

$$A_{1,\tau} = k_1^2 A_1 - \mathrm{i}4k_1^2 \int_{-\infty}^{\infty} \delta(\Omega) k' A_1(k') A_1(k_1 - k') \mathrm{e}^{\mathrm{i}\Omega(k_1,k')t}\, \mathrm{d}k'$$

$$= k_1^2 A_1 - \mathrm{i}4k_1^2 \sum_{\Omega(k')=0} k' A_1(k') A_1(k_1 - k'). \qquad \text{(A 9)}$$

This is a resonance equation to determine the amplitude-density function A_1, and the continuous integral is replaced by the summation of many discrete resonant trios. It can be shown that multiple-wave resonance exists by including more terms in (A 5). Also, it is clear that a substantial amount of energy can only transfer among waves via resonance.

The conditions for resonant trios are that the wavenumbers $k', k_1 - k', k_1$ form a closed polygon and the corresponding frequency relation, $\Omega(k') = 0$. The roots of the algebraic resonance condition, $\Omega = 0$ are discrete wavenumbers. This is an important property and implies that energy transfers among waves discretely. Although we cannot rigorously prove this statement for a general case, physical arguments as well as numerical evidence support it.

The roots of the algebraic resonance conditions form a denumerable set. The ratios of wavenumbers can be either rational numbers or irrational numbers. For irrational ratios, the solution is homeomorphic to Cantor's set of irrational numbers, and is topologically transitive in x space. However, with the exception of the case of a constant wave speed, we cannot find incommensurate waves satisfying the resonance conditions for many cases of well known or simple frequency–wavenumber relations. This seems to imply that no energy can transfer resonantly among incommensurate waves. At present, we cannot prove rigorously that this is true as a general rule. It may require an extensive search before a definite conclusion can be made. It is interesting to note that an important consequence of commensurate waves is the self-similarity of strange attractors which is not topologically transitive.

For commensurate waves, the set of resonant waves is homeomorphic to Cantor's set of rational numbers, and forms a closed multi-dimensional torus; equivalently, the waves are mode locked and in 'Arnold tongues' (Arnold 1965). Since infinitely many such sets can exist in a finite real interval and initial conditions determine which set will be excited, nonlinear partial differential equations can have multiple equilibrium solutions. The equilibrium solution is defined as a long-time averaged solution here. Almost identical initial conditions with small difference that can hardly be noticed in a normal plot can result in different solutions. This means that the solution is sensitive to initial conditions as described by chaos theories if the Liapunov exponent is positive. The case that a long wave can excite a linear unstable short wave through a sequence of harmonics (Yao & Ghosh Moulic 1995*b*) is a good example to show how and why a solution is sensitive to the initial condition. The final equilibrium solution of various transient solutions due to different initial conditions can be identical within each set. On the other hand, the solutions from different sets will have completely different equilibrium solutions with almost identical initial conditions.

The phase angles of waves are modified by resonance as shown in (A 9). Even though the frequency associated with a wave is discrete at any instant, a time-averaged measurement will record a continuous frequency due to the resonance broadening effect. The resonance condition for the Kuramoto–Sivashinsky equation near the first bifurcation point is trivial, since there are stationary waves. However, I will show from the following numerical results that travelling waves develop when the value of λ increases.

A pseudospectral method is used to spatially discretize (A 1) for x in the interval $[0, 2\pi]$ (Jolly, Kevrekidis & Titi 1990). The excited set is therefore implicitly selected. The solution is written as

$$u(x,t) = \sum_{n=1}^{\frac{1}{2}N-1} (\hat{u}_n e^{inx} + \text{c.c.}), \tag{A 10}$$

where c.c. denotes the complex conjugate. The wavenumbers are commensurate, but are dense. This is an unavoidable limitation of numerical solutions. A time-splitting scheme is used for the temporal discretization. The nonlinear term is approximated by an explicit second-order-accurate Adams–Bashforth scheme and the linear term by an implicit Crank–Nicholson scheme.

For λ equal to 69, four modes are linearly unstable. The converged numerical results presented below are for $N = 64$, and $\Delta t = 0.0002$. The amplitude of the wave $n = 2$ is set at 1 initially. It is clear from figure 1(*a*) that the even waves are split into two groups at about time step 20. The group headed by the wave $n = 2$ joins with the group of odd waves at about time step 110. The complex interaction begins at about time step 180 when the amplitudes of the leading members of this joint group become comparable to the originally dominant group led by the wave $n = 4$. The energy transfers resonantly among four groups led by waves $n = 1,\ 2,\ 3,$ and 4, respectively. The solution is not periodic in time and very complex. This is because that the wave frequencies are modified continuously by resonance. An amplified plot

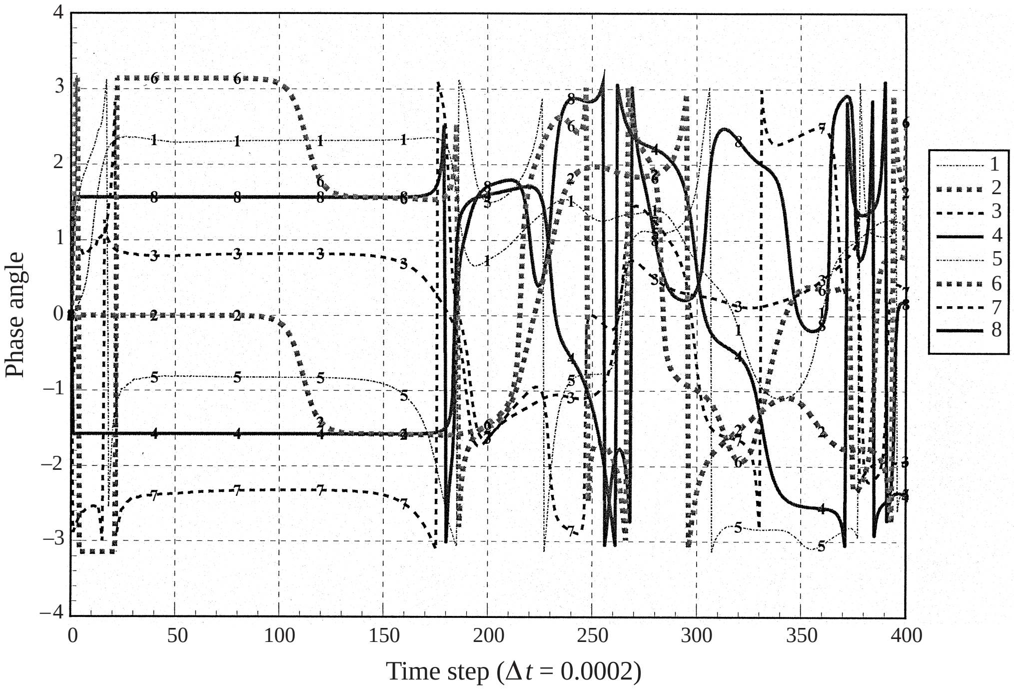

FIGURE 2. Phase angle with initial mode: $2 = (1, 0)$.

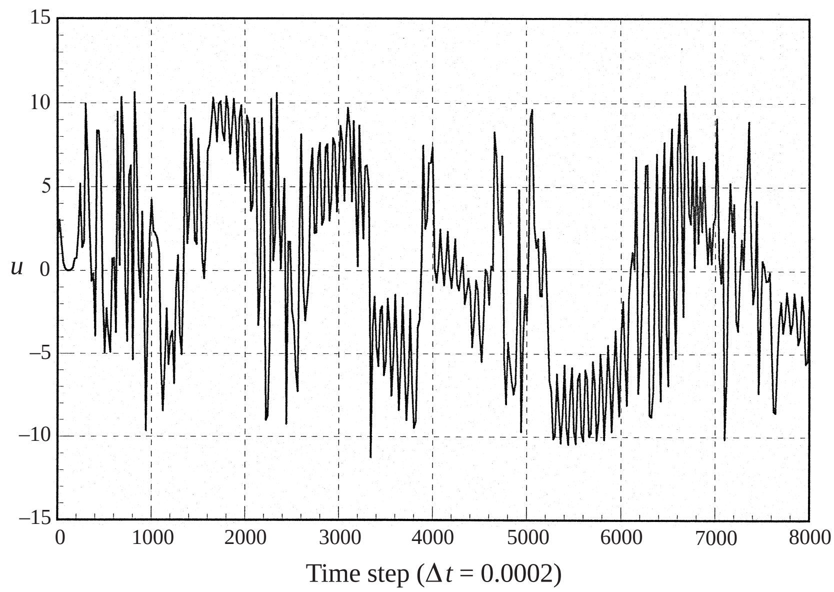

FIGURE 3. Time history of u at $x = 0$.

of first eight waves in figure 1(*b*) shows the waves within each group have their own detailed evolution. This indicates that resonance can occur independently among waves from different groups. A long-time plot of the first four waves in figure 1(*c*) shows that the solution has never become periodic.

The phase angles in $[-\pi, \pi]$ of the first eight waves are plotted in figure 2. The time derivative of the phase angle is the wave frequency. The wave frequency is discrete at any instant time, but continuously modified by resonance. The resonance broadening effect on frequency for a time-averaged measurement is clearly demonstrated in the figure. It also shows that the frequency of waves is not necessarily a monotonic function of the wavenumber! The time-history of the function u is given in figure 3. The numerical results show that the source of complex unsteady solutions of the Kuramoto–Sivashinsky equation can be due to many waves exchanging energy resonantly and never reaching equilibrium.

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